



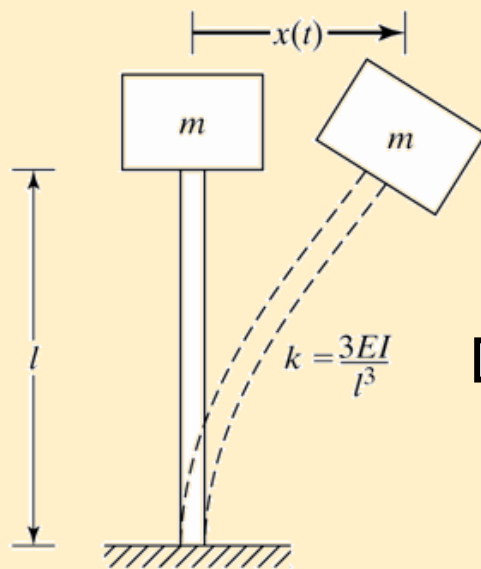
Mechanical Vibrations

1

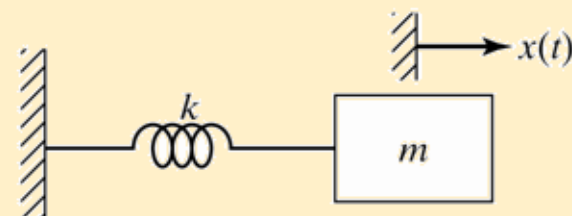


Free vibration of single degree of freedom system

Examples



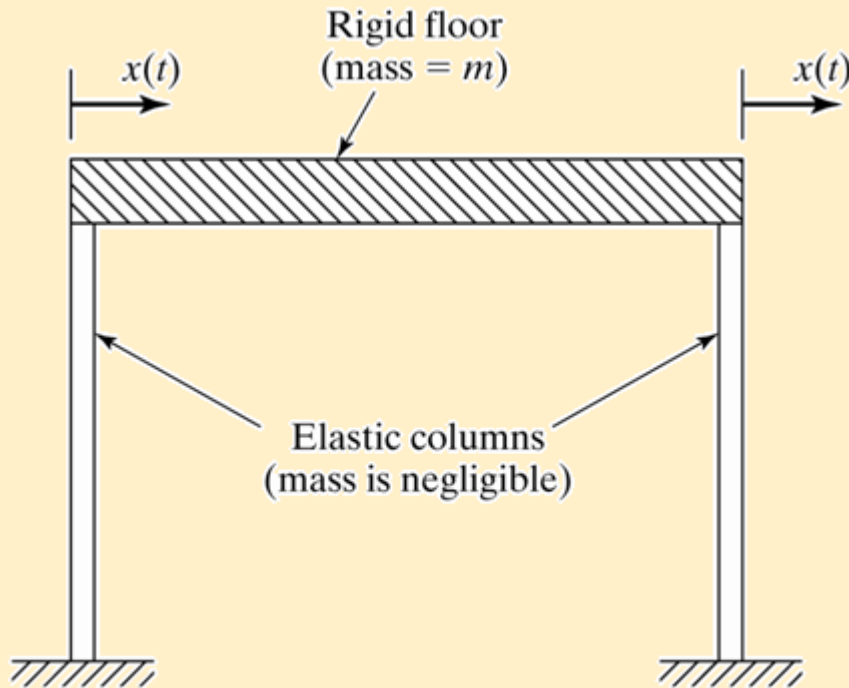
(a) Idealization of the tall structure



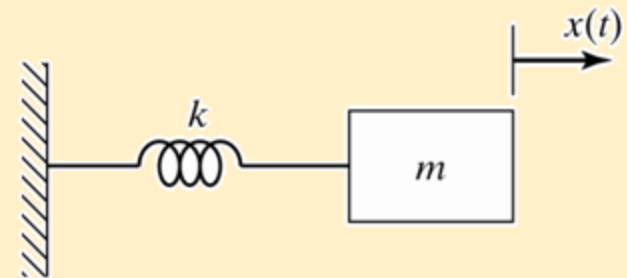
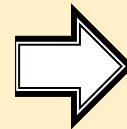
(b) Equivalent spring-mass system

Free vibration of single degree of freedom system

Examples

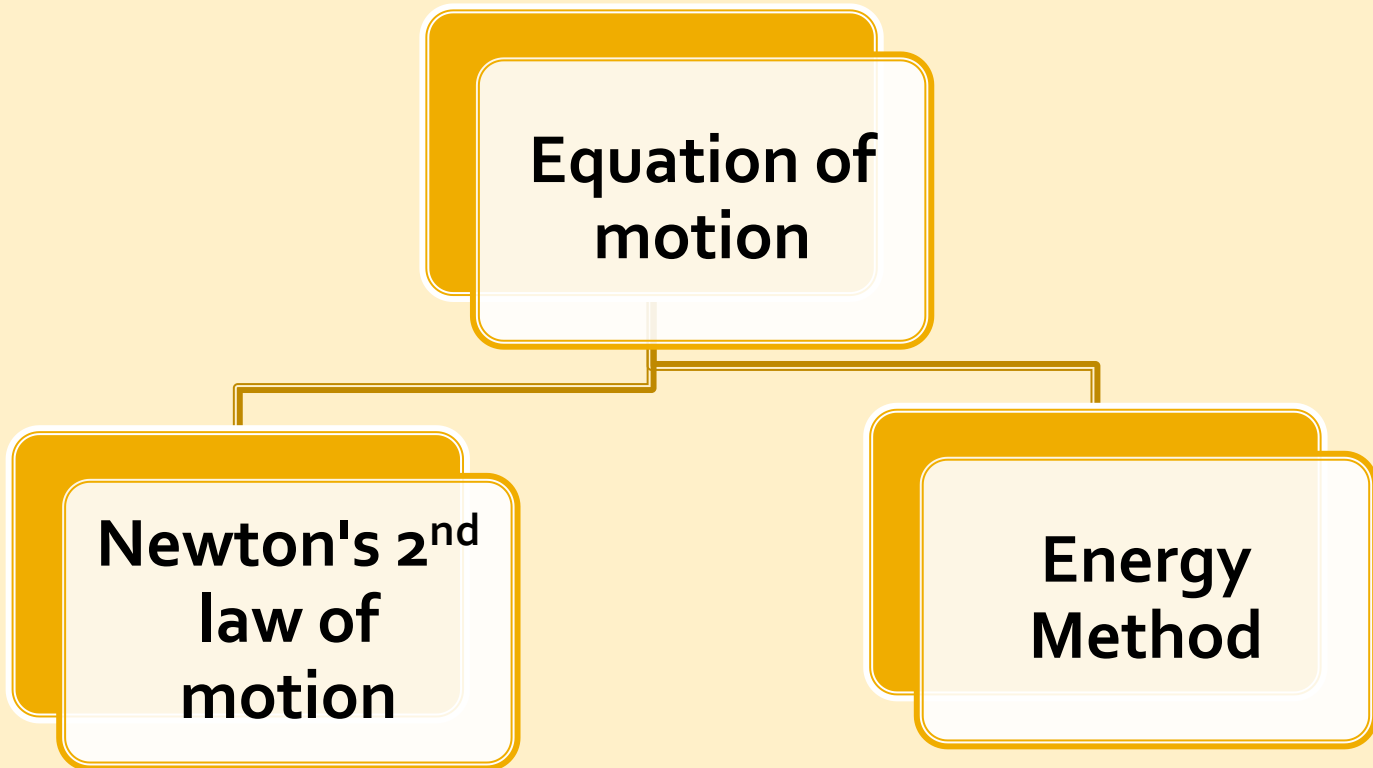


(a) Building frame



Free vibration of single degree of freedom system

□ Equation of motion



Free vibration of single degree of freedom system

□ Newton's 2nd law of motion

Select a
suitable
coordinate

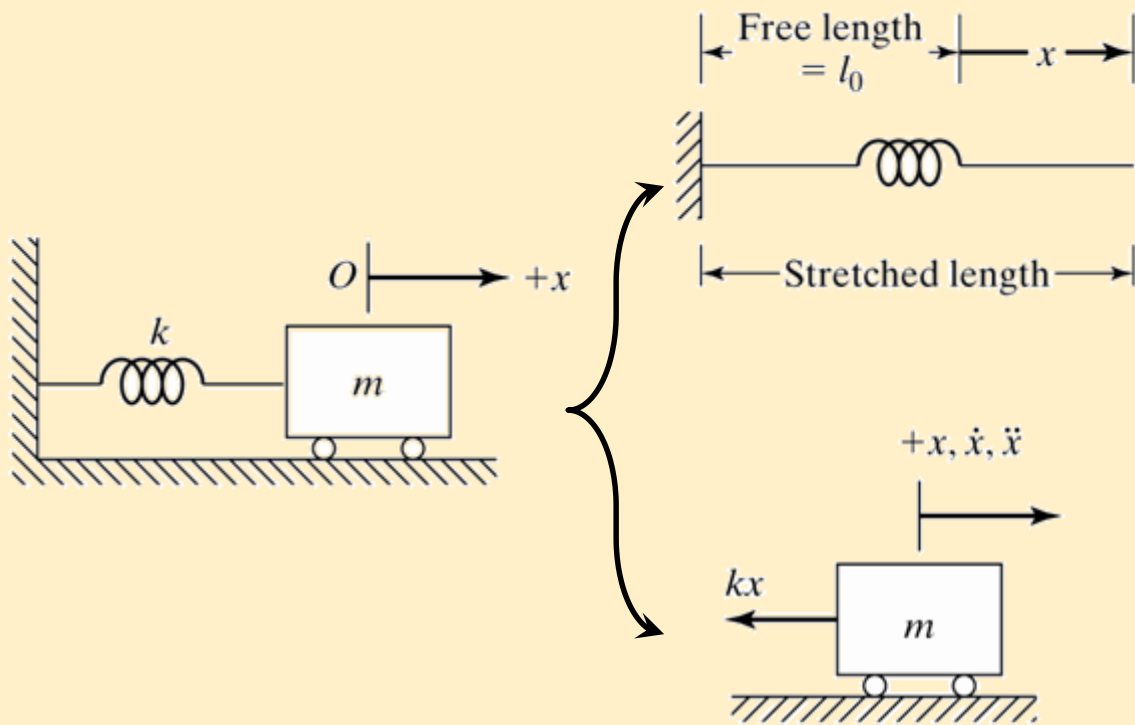
Determine the static
equilibrium configuration
of the system

Draw the
free-body
diagram

Apply
Newton's
second law
of motion

Free vibration of single degree of freedom system

Example

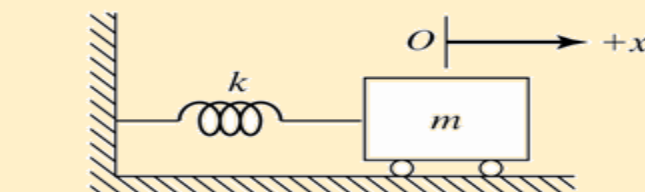


Free vibration of single degree of freedom system

□ Newton's 2nd law of motion

$$\sum F = m a$$

$$-kx = m \ddot{x} \Rightarrow m \ddot{x} + kx = 0$$



homogeneous 2nd order ODE

$$\ddot{x} + \omega_n^2 x = 0, \quad \omega_n = \sqrt{\frac{k}{m}}$$

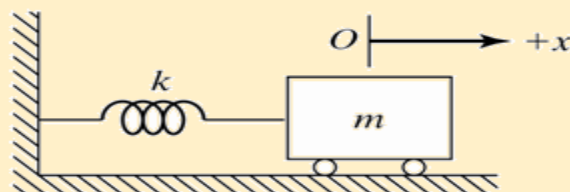
$$x(t) = Be^{st}$$

$$(s^2 + \omega_n^2)Be^{st} = 0 \quad \xrightarrow{Be^{st} \neq 0} \quad s^2 + \omega_n^2 = 0 \quad \rightarrow \quad s = \pm i \omega_n$$

$$\rightarrow x(t) = B_1 e^{i \omega_n t} + B_2 e^{-i \omega_n t}$$

Free vibration of single degree of freedom system

□ Newton's 2nd law of motion



$$x(t) = B_1 e^{i\omega_n t} + B_2 e^{-i\omega_n t} = B_1 (\cos(\omega_n t) + i \sin(\omega_n t)) + B_2 (\cos(\omega_n t) - i \sin(\omega_n t))$$

$$x(t) = (B_1 + B_2) \cos(\omega_n t) + i(B_1 - B_2) \sin(\omega_n t)$$

$$B_1 = \bar{B}_2 \rightarrow (B_1 + B_2) = 2 \operatorname{Re}\{B_1\} = A_1$$

$$B_1 = \bar{B}_2 \rightarrow (B_1 - B_2) = 2i \operatorname{Im}\{B_1\} = -A_2$$

$$x(t) = A_1 \cos(\omega_n t) + A_2 \sin(\omega_n t)$$

$$A_1 = A \sin(\theta), A_2 = A \cos(\theta)$$

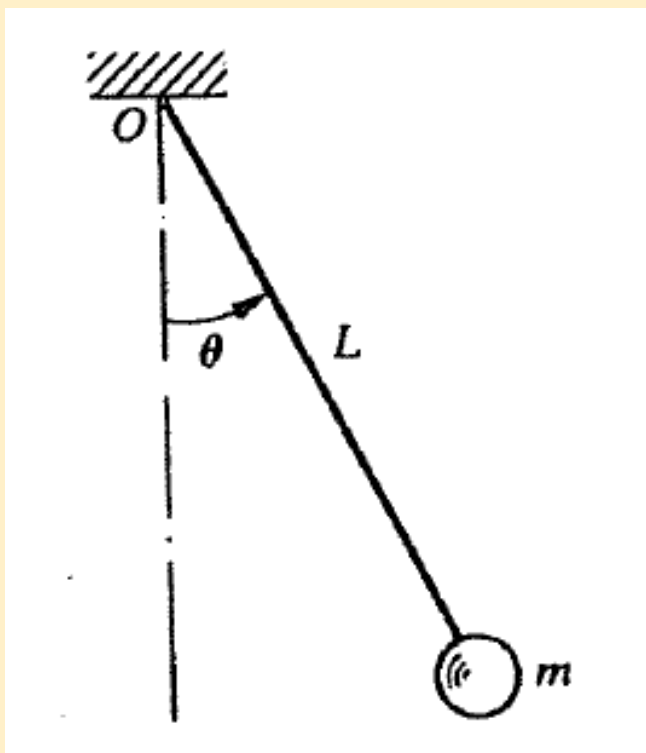
$$\rightarrow x(t) = A \sin(\theta) \cos(\omega_n t) + A \cos(\theta) \sin(\omega_n t)$$

$$= A \sin(\omega_n t + \theta)$$

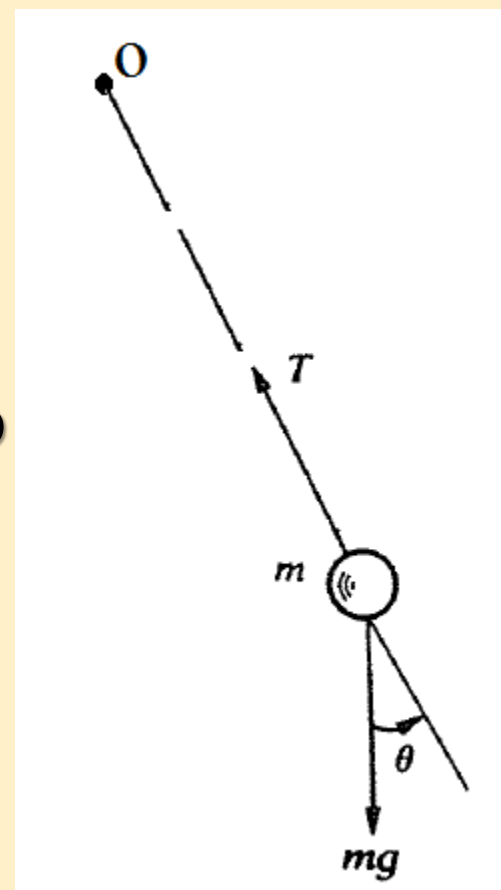
Free vibration of single degree of freedom system

Example.

Find the equation of motion.



F.B.D



Free vibration of single degree of freedom system

$$\sum M_o = I_o \alpha \quad \rightarrow \quad -mgL \sin(\theta) = (mL^2) \ddot{\theta}$$

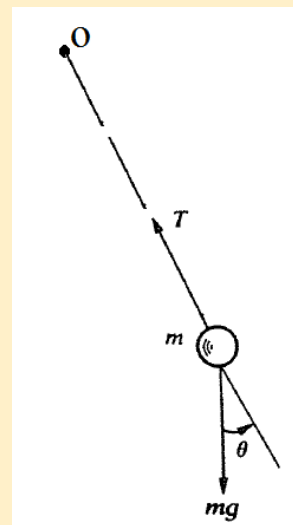
$$\rightarrow mL^2 \ddot{\theta} + mgL \sin(\theta) = 0$$

$$\rightarrow L \ddot{\theta} + g \sin(\theta) = 0$$

Non-linear 2nd-Order ODE

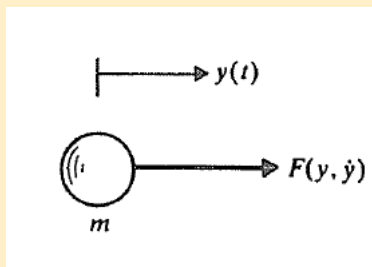
Solving the Non-linear ODE

Small motion assumptions (Linearization)



Free vibration of single degree of freedom system

Linearization



$$\sum F = Ma \rightarrow F(y, \dot{y}) = m\ddot{y} \rightarrow \ddot{y} = \frac{1}{m} F(y, \dot{y}) = f(y, \dot{y})$$

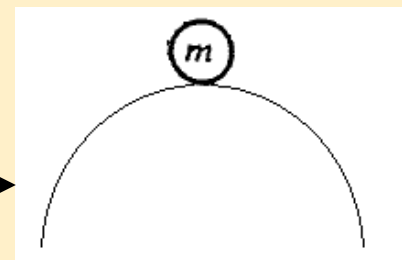
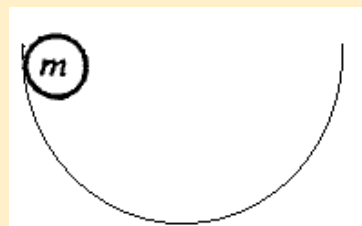
$$y = y_e = \text{constant}, \dot{y} = \ddot{y} = 0$$

$$f(y_e, 0) = 0$$

• Asymptotically stable

• Stable

• Unstable



Free vibration of single degree of freedom system

Linearization

$$f(y, \dot{y}) = f(y_e, 0) + \left. \frac{\partial f(y, \dot{y})}{\partial y} \right|_e (y - y_e) + \left. \frac{\partial f(y, \dot{y})}{\partial \dot{y}} \right|_e \dot{y} + O(y, \dot{y})$$

$$\left. \frac{\partial f(y, \dot{y})}{\partial y} \right|_e = -b \quad \left. \frac{\partial f(y, \dot{y})}{\partial \dot{y}} \right|_e = -a$$

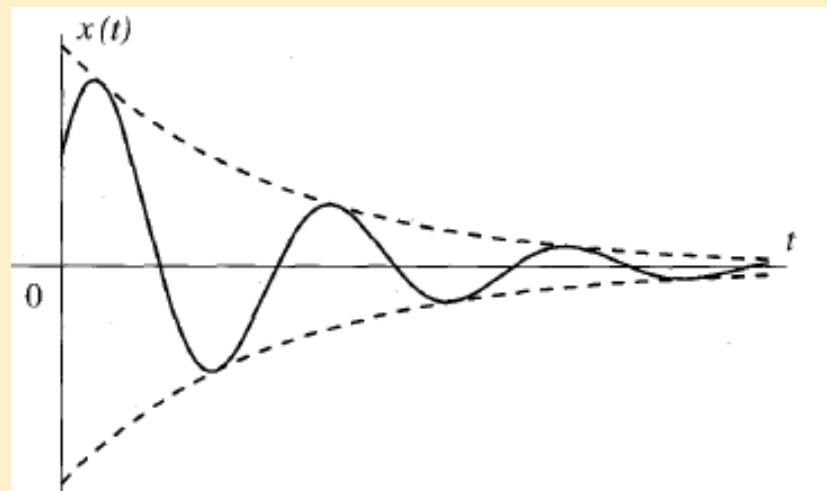
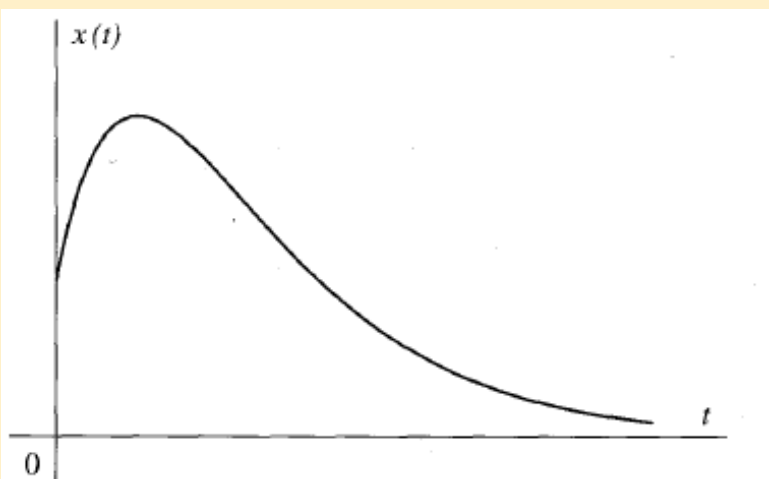
$$\rightarrow \ddot{y} = -b(y - y_e) - a\dot{y} \xrightarrow{x=(y-y_e)} \ddot{x} + a\dot{x} + bx = 0$$

$$\text{Sol.: } x(t) = Ae^{st} \longrightarrow s^2 + as + b = 0$$

$$\begin{matrix} s_1 \\ s_2 \end{matrix} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \longrightarrow x(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

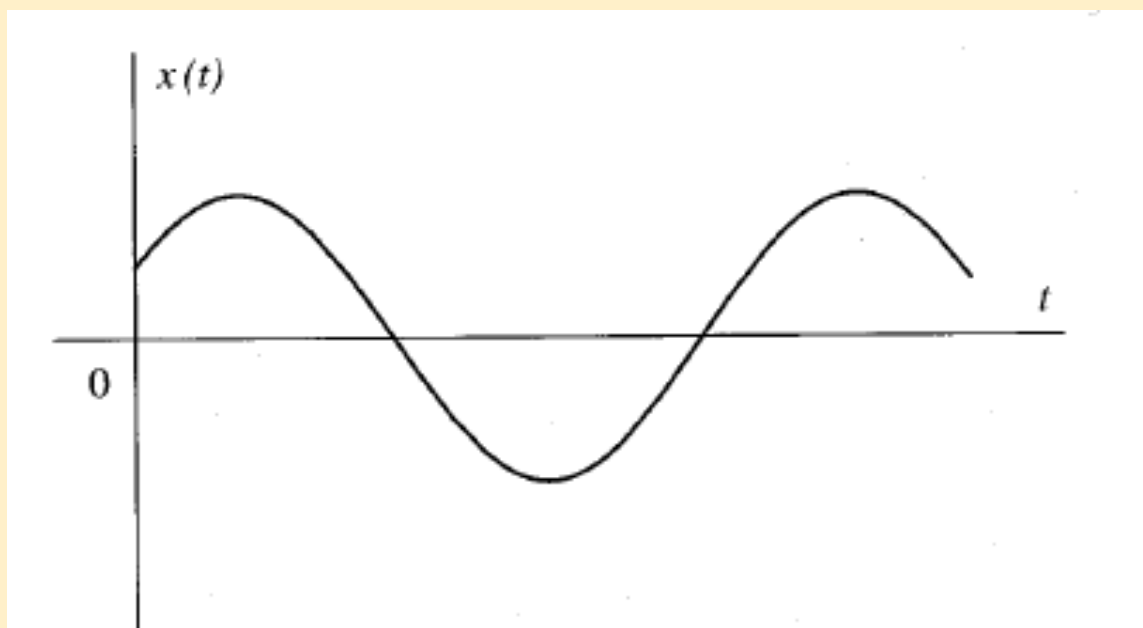
Free vibration of single degree of freedom system

1. $a > 0$, $b > 0$. In this case the roots are either real and negative or complex conjugates with negative real part, so that $x(t)$ approaches zero as $t \rightarrow \infty$. Hence, $y(t)$ approaches y_e , so that the equilibrium position is asymptotically stable.



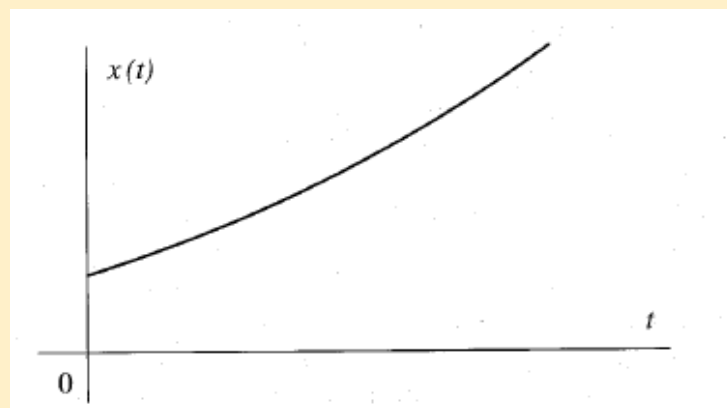
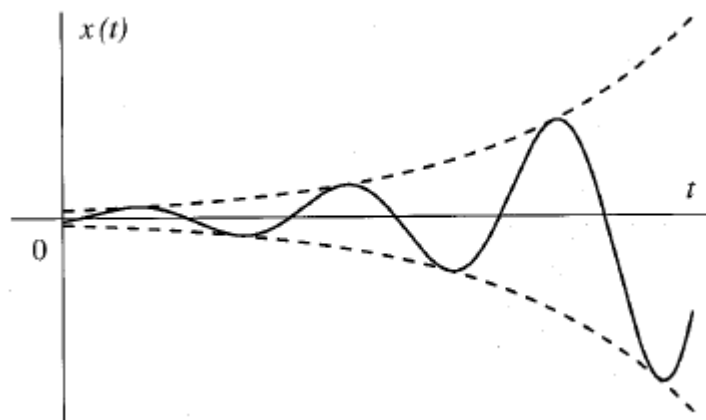
Free vibration of single degree of freedom system

2. $a = 0$, $b > 0$. The roots are pure imaginary, so that the solution $x(t)$ is oscillatory. Hence, the motion is bounded and the equilibrium position is stable.



Free vibration of single degree of freedom system

3. $a < 0$, or $a \geq 0$, $b < 0$. The roots are either complex conjugates with positive real part or they are both real, with one root being positive and the other being negative. In either case the solution is divergent and the equilibrium position is unstable.



Free vibration of single degree of freedom system

Return to example:

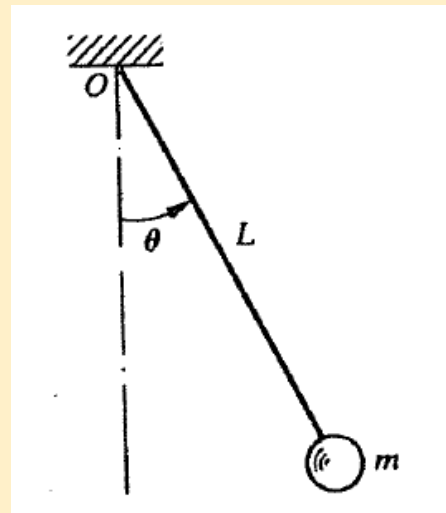
$$L \ddot{\theta} + g \sin(\theta) = 0 \quad \rightarrow \quad \ddot{\theta} = -\frac{g}{L} \sin(\theta)$$

$$\ddot{\theta} = f(\theta), \quad f(\theta) = -\frac{g}{L} \sin \theta$$

$$f(\theta_e) = -\frac{g}{L} \sin \theta_e = 0$$

$$\theta_e = 0, \pm\pi, \pm2\pi, \dots$$

$$\theta_{e1} = 0 \quad \theta_{e2} = \pi$$



Free vibration of single degree of freedom system

$$\ddot{\theta} = f(\theta) \quad \underset{0}{=} f(\theta_e) + \left. \frac{\partial f(\theta)}{\partial \theta} \right|_{\theta=\theta_e} (\theta - \theta_e) + o(\theta)$$

$$x = \theta - \theta_e \quad \rightarrow \quad \ddot{x} = -bx \quad \rightarrow \quad \ddot{x} + bx = 0$$

$$b = -\left. \frac{\partial f}{\partial \theta} \right|_{\theta=\theta_e} = \frac{g}{L} \cos \theta_e$$

$$\theta_{e1} = 0$$

$$b = \frac{g}{L} > 0$$

so that the equilibrium is stable

$$\theta_{e2} = \pi$$

$$b = -\frac{g}{L} < 0$$

so that the equilibrium is unstable

Free vibration of single degree of freedom system

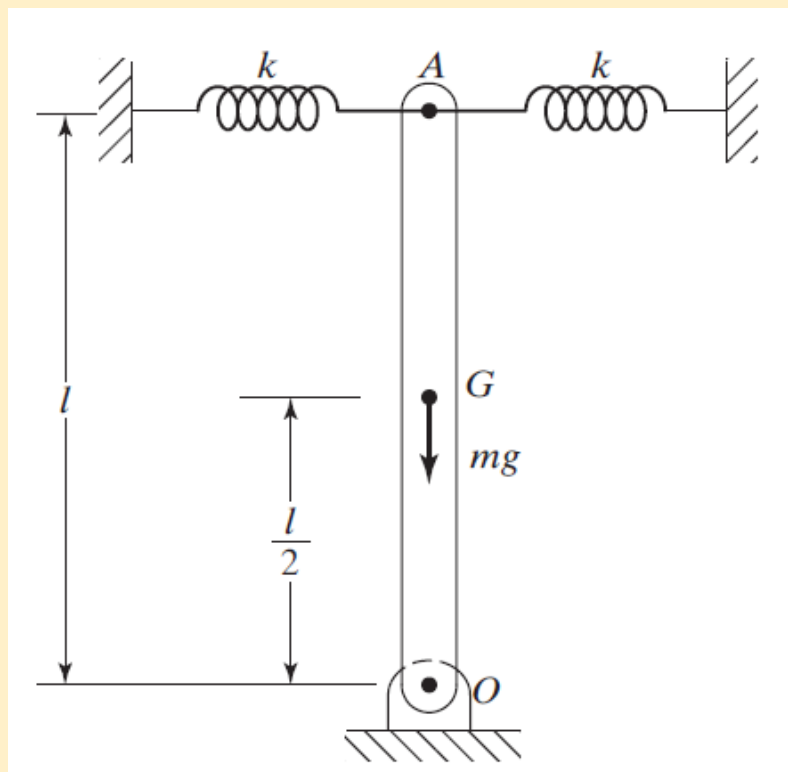
$$L \ddot{\theta} + g \sin(\theta) = 0 \quad \xrightarrow{\text{Linearization about } 0^\circ} \ddot{\theta} + \frac{g}{L} \theta = 0$$

In Summary: *if* $\theta < 6^\circ \xrightarrow{\text{Linearization about } 0^\circ} \begin{cases} \sin(\theta) \simeq \theta \\ \cos(\theta) \simeq 1 \end{cases}$

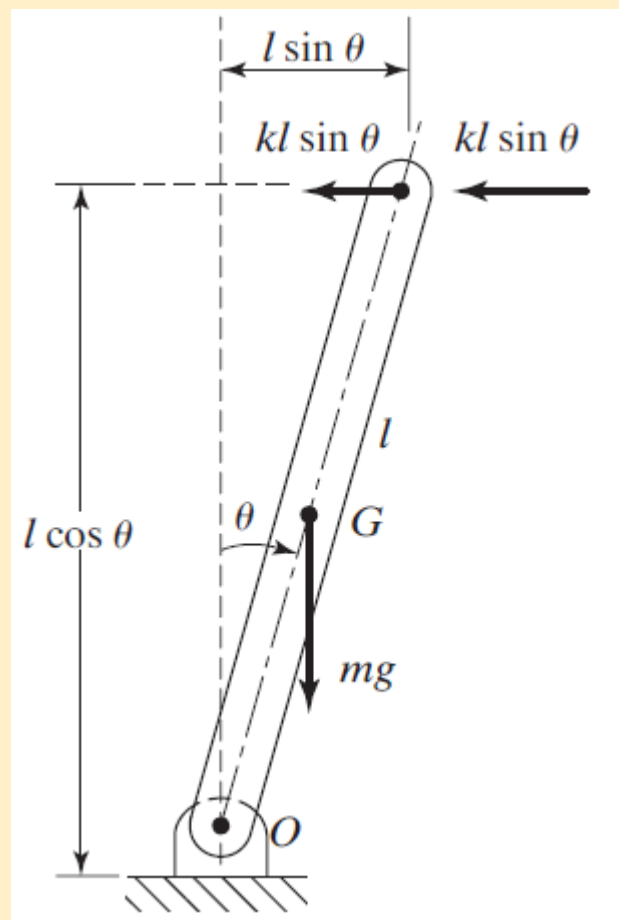
Free vibration of single degree of freedom system

Example.

Find the equation of motion



F.B.D

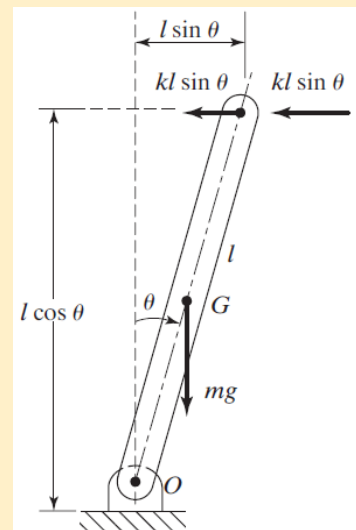


Free vibration of single degree of freedom system

$$\sum M_o = I_o \alpha$$

$$\rightarrow W \frac{l}{2} \sin(\theta) - (2kl \sin(\theta)) l \cos(\theta) = \left(\frac{1}{3} ml^2 \right) \ddot{\theta}$$

$$\rightarrow \frac{ml^2}{3} \ddot{\theta} + (2kl \sin \theta) l \cos \theta - W \frac{l}{2} \sin \theta = 0$$

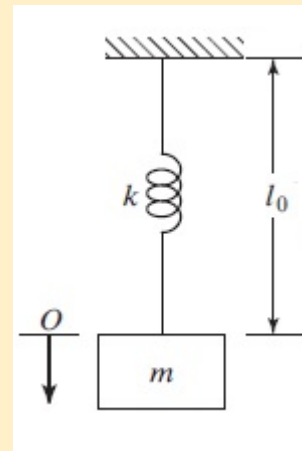


For small oscillations.

$$\rightarrow \frac{ml^2}{3} \ddot{\theta} + 2kl^2 \theta - \frac{Wl}{2} \theta = 0$$

$$\Rightarrow \frac{ml^2}{3} \ddot{\theta} + \left(2kl^2 - \frac{mgl}{2} \right) \theta = 0$$

Free vibration of single degree of freedom system



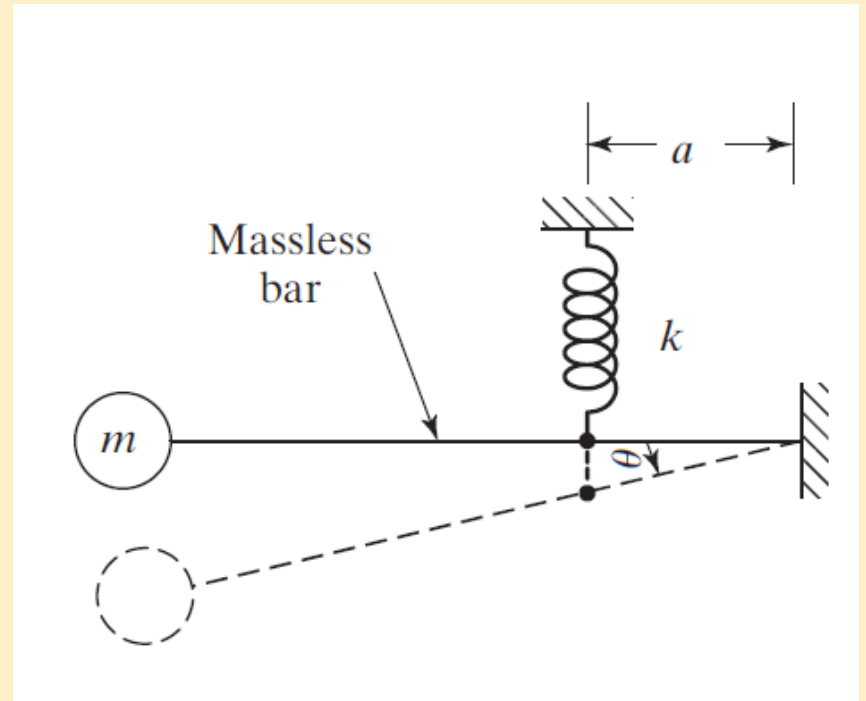
$$\sum F = ma \rightarrow mg - kx = m\ddot{x} \rightarrow m\ddot{x} + kx = mg$$

$$\Rightarrow x = x_h + x_p = A \sin(\omega_n t + \phi) + \frac{mg}{k}$$

$$x = y + \delta_{st} \rightarrow \ddot{x} = \ddot{y} \rightarrow m\ddot{y} + k(y + \delta_{st}) = mg$$

$$\Rightarrow m\ddot{y} + ky = 0 \rightarrow y = A \sin(\omega_n t + \phi)$$

Free vibration of single degree of freedom system



$$\sum M_c = I_c \alpha \rightarrow mgL \cos \theta - ka^2 \theta \cos \theta = mL^2 \ddot{\theta} \rightarrow$$

$$\theta < 6^\circ \rightarrow mL^2 \ddot{\theta} + ka^2 \theta = mgL$$

Free vibration of single degree of freedom system

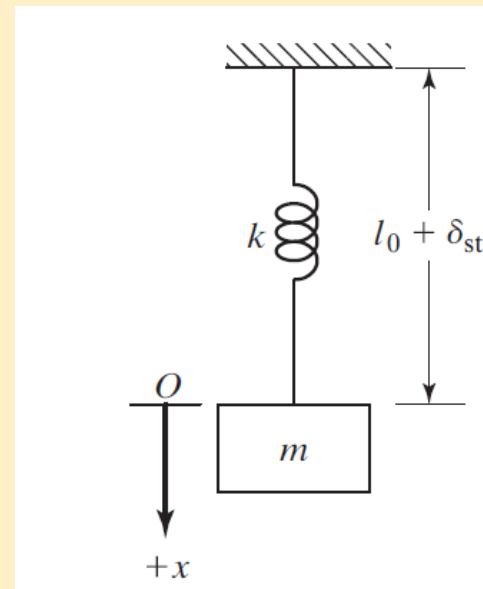
■ Energy Method

$$E_1 = E_2 \rightarrow T_1 + U_1 = T_2 + U_2$$

if 1: static equilibrium $\rightarrow T_1 = 0, U_1 = 0$ or constant

$$\rightarrow T + U = \text{constant}$$

$$\Rightarrow \frac{d}{dt}(T + U) = 0$$



Free vibration of single degree of freedom system

■ Energy Method

$$\overline{dW} = \mathbf{F} \cdot d\mathbf{r}$$

$$\overline{dW} = \mathbf{F} \cdot d\mathbf{r} = m\ddot{\mathbf{r}} \cdot d\mathbf{r} = m \frac{d\dot{\mathbf{r}}}{dt} \cdot \frac{d\mathbf{r}}{dt} dt$$

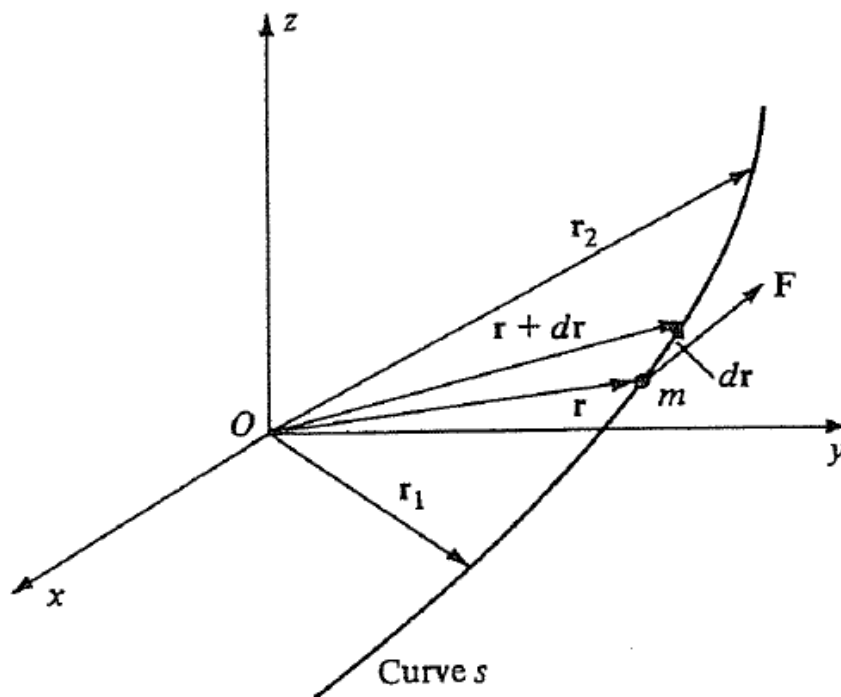
$$= m\dot{\mathbf{r}} \cdot d\dot{\mathbf{r}} = d\left(\frac{1}{2}m\dot{\mathbf{r}} \cdot \dot{\mathbf{r}}\right) = dT$$

Conservative fore:

$$dW = \mathbf{F} \cdot d\mathbf{r} = -dV(\mathbf{r})$$

$$dT = -dV \rightarrow d(T + V) = 0$$

$$\rightarrow T + V = \text{cte.}$$



Free vibration of single degree of freedom system

■ Energy Method

Non-Conservative fore: $\mathbf{F} = \mathbf{F}_c + \mathbf{F}_{nc}$

$$\overline{dW} = \mathbf{F} \cdot d\mathbf{r}$$

$$\rightarrow \overline{dW} = -dV + F_{nc} \cdot dr$$

$$\rightarrow dT = -dV + F_{nc} \cdot dr$$

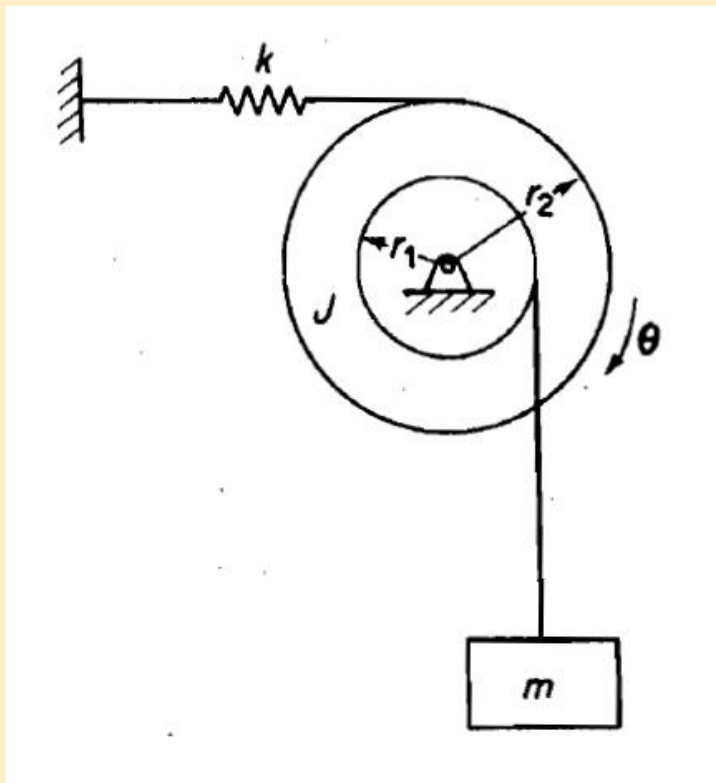
$$\rightarrow d(T + V) = F_{nc} \cdot dr$$

$$\rightarrow \frac{d}{dt}(T + V) = F_{nc} \cdot \dot{r}$$

Free vibration of single degree of freedom system

Example:

Find the equation of motion.



$$mg * r_1 = kr_2 \theta_{st} * r_2$$

Free vibration of single degree of freedom system

$$\frac{d}{dt}(T + U) = 0$$

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2 \quad U = \frac{1}{2} k y^2$$

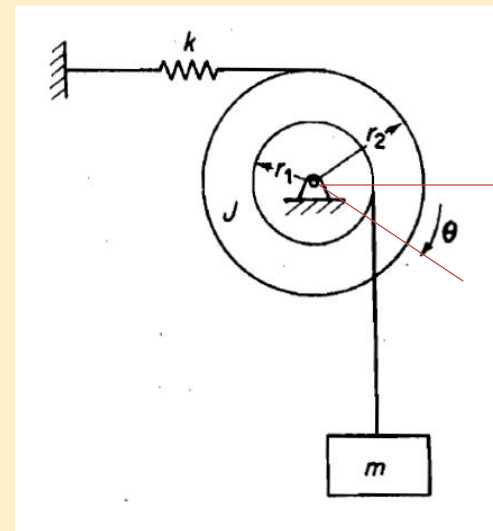
$$x = r_1 \theta, y = r_2 \theta$$

$$T = \frac{1}{2} m (r_1 \dot{\theta})^2 + \frac{1}{2} J \dot{\theta}^2, \quad U = \frac{1}{2} k (r_2 \theta)^2$$

$$\frac{d}{dt} \left(\left(\frac{1}{2} m (r_1 \dot{\theta})^2 + \frac{1}{2} J \dot{\theta}^2 \right) + \left(\frac{1}{2} k (r_2 \theta)^2 \right) \right) = 0$$

$$m r_1^2 \ddot{\theta} + J \ddot{\theta} + k r_2^2 \theta = 0 \quad \xrightarrow{\dot{\theta} \neq 0} m r_1^2 \ddot{\theta} + J \ddot{\theta} + k r_2^2 \theta = 0$$

$$\Rightarrow (m r_1^2 + J) \ddot{\theta} + k r_2^2 \theta = 0$$



Free vibration of single degree of freedom system

■ Maximum Energy Method

$$E_1 = E_2 \rightarrow T_1 + U_1 = T_2 + U_2$$

1: Max(T) , 2: Max(U)

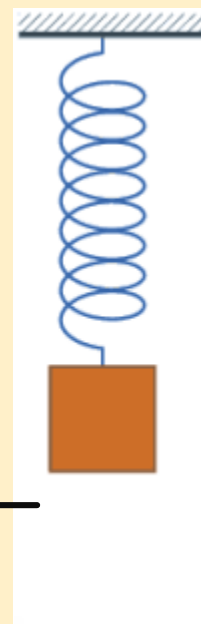
$$T_1 + 0 = 0 + U_2 \longrightarrow T_{Max} = U_{Max}$$

$$\frac{1}{2} m \dot{x}_{Max}^2 = \frac{1}{2} k x_{Max}^2 \quad (1)$$

$$x(t) = A \sin(\omega_n t) \rightarrow x_{Max} = A$$

$$\dot{x}(t) = A \omega_n \cos(\omega_n t) \rightarrow \dot{x}_{Max} = A \omega_n \quad \left. \vphantom{\dot{x}(t)} \right\} \dot{x}_{Max} = \omega_n x_{Max}$$

$$(1) \rightarrow m (\omega_n x_{Max})^2 = k x_{Max}^2 \rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

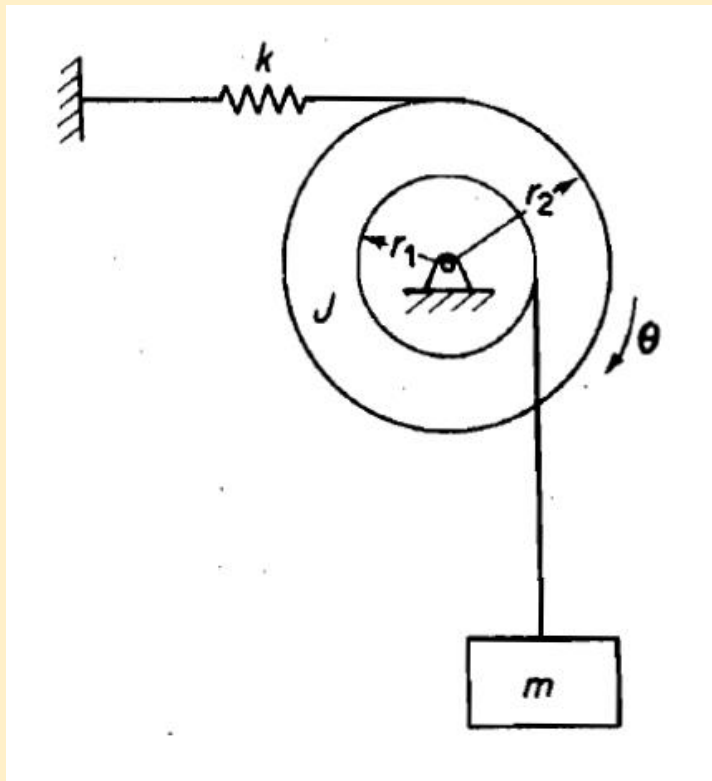


$$\omega_n$$

Free vibration of single degree of freedom system

Example:

Find the natural frequency.



Free vibration of single degree of freedom system

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2 \quad U = \frac{1}{2} k y^2$$

$$x = r_1 \theta, y = r_2 \theta$$

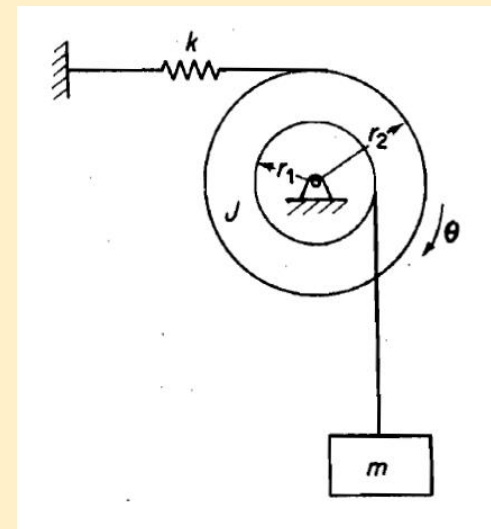
$$T_{Max} = \frac{1}{2} m (r_1 \dot{\theta}_{Max})^2 + \frac{1}{2} J \dot{\theta}_{Max}^2, \quad U_{Max} = \frac{1}{2} k (r_2 \theta_{Max})^2$$

$$d(T + U) = 0 \rightarrow T_{Max} = U_{Max}$$

$$\frac{1}{2} (m r_1^2 + J) \dot{\theta}_{Max}^2 = \frac{1}{2} (k r_2^2) \theta_{Max}^2$$

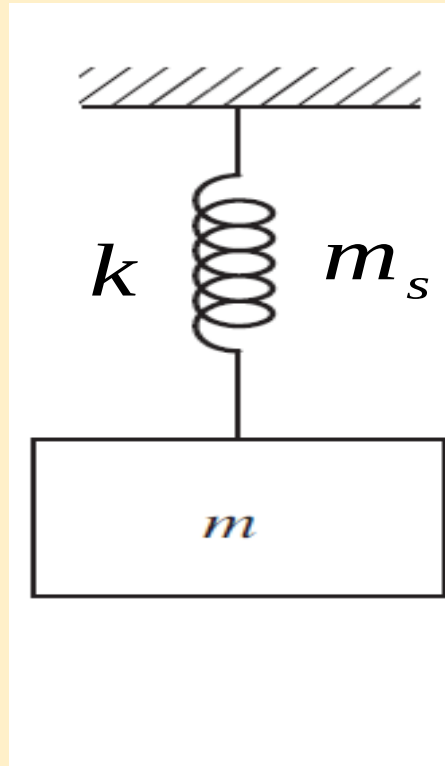
$$\xrightarrow{\dot{\theta}_{Max} = \omega_n \theta_{Max}} \omega_n = \sqrt{\frac{k r_2^2}{m r_1^2 + J}}$$

$$m_{eff} = m r_1^2 + J, \quad k_{eff} = k r_2^2 \rightarrow \omega_n = \sqrt{\frac{k_{eff}}{m_{eff}}} \rightarrow m_{eff} \ddot{\theta} + k_{eff} \theta = 0$$



Free vibration of single degree of freedom system

Rayleigh s Method (Effective Mass)



Free vibration of single degree of freedom system

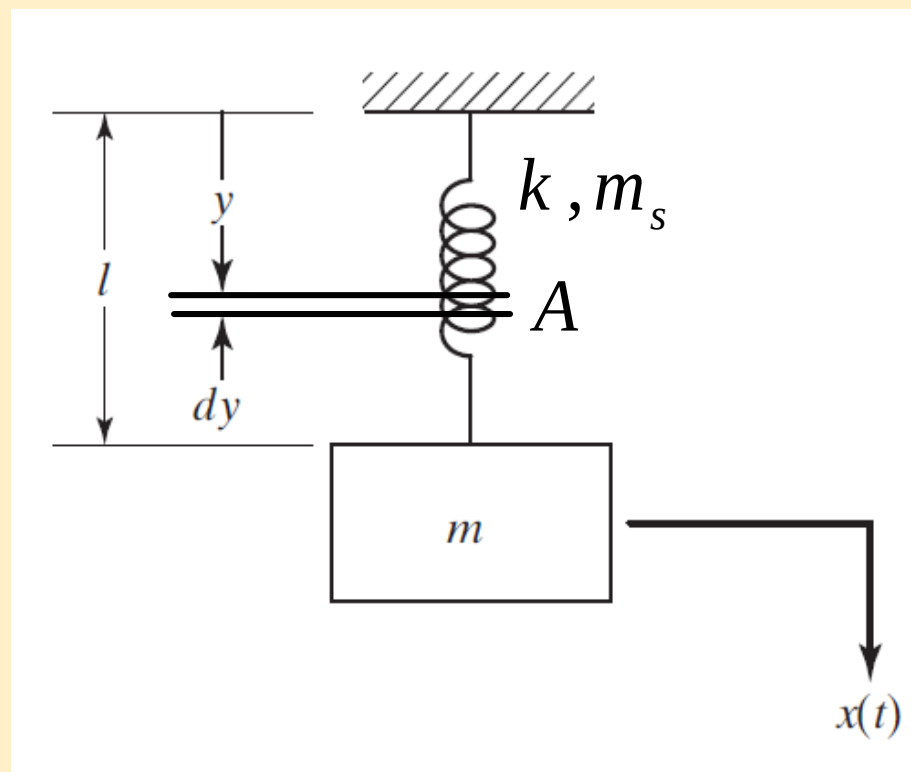
Rayleigh's Method (Effective Mass)

$$T_{total} = \frac{1}{2} m \dot{x}^2 + T_s(\dot{x})$$

$$Z_A = \frac{x}{l} y \rightarrow Z_A = \frac{y}{l} x$$

$$\rightarrow V_A = \frac{y}{l} \dot{x}$$

$$\begin{aligned} \rightarrow dT_s &= \frac{1}{2} \left(\frac{m_s}{l} dy \right) (v_A)^2 \\ &= \frac{1}{2} \left(\frac{m_s}{l} dy \right) \left(\frac{y}{l} \dot{x} \right)^2 \\ &= \frac{1}{2} \left(\frac{m_s}{l^3} y^2 dy \right) \dot{x}^2 \end{aligned}$$



Free vibration of single degree of freedom system

Rayleigh's Method (Effective Mass)

$$T_s = \int_0^l \frac{1}{2} \left(\frac{m_s}{l^3} y^2 dy \right) \dot{x}^2 = \frac{1}{2} \left(\frac{m_s}{3} \right) \dot{x}^2$$

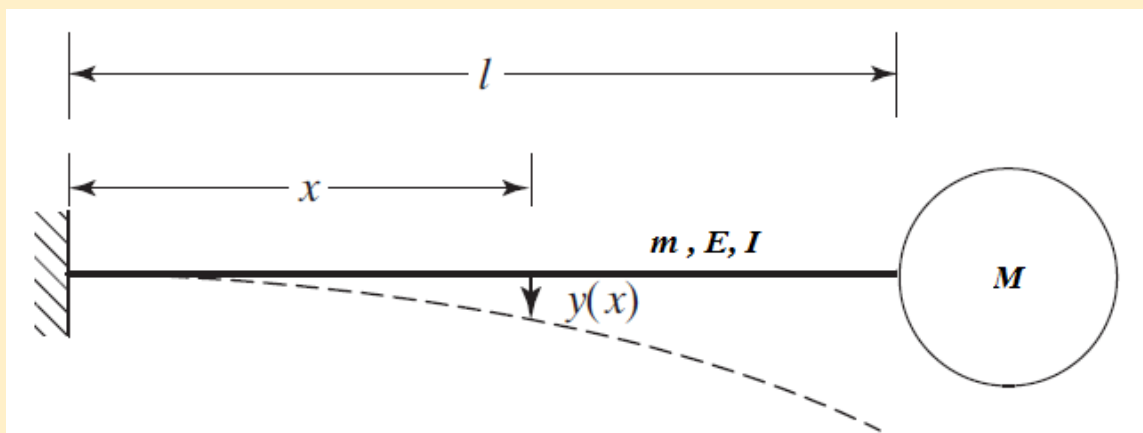
Effective mass of spring

$$\begin{aligned} \Rightarrow T_{total} &= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} \frac{m_s}{3} \dot{x}^2 \\ &= \frac{1}{2} \left(m + \frac{m_s}{3} \right) \dot{x}^2 \end{aligned}$$

Free vibration of single degree of freedom system

Example.

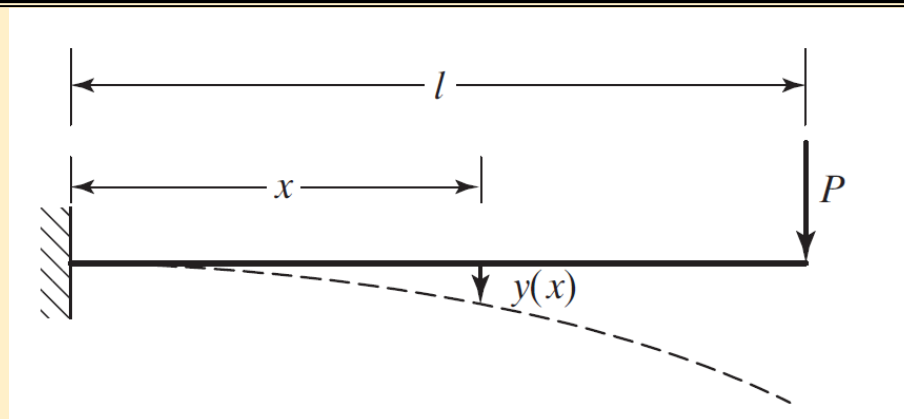
Find the Natural Frequency.



$$y(x) = \frac{Px^2}{6EI}(3l - x)$$

Free vibration of single degree of freedom system

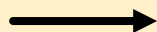
Sol.



$$y_{\max} = \frac{Pl^3}{3EI}$$



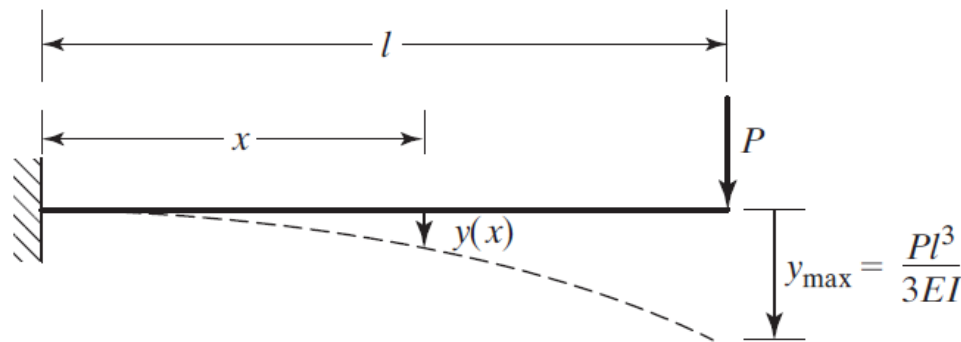
$$\begin{aligned} y(x) &= \frac{Px^2}{6EI}(3l - x) = \frac{y_{\max}x^2}{2l^3}(3l - x) \\ &= \frac{y_{\max}}{2l^3}(3x^2l - x^3) \end{aligned}$$



$$\dot{y}(x) = \frac{\dot{y}_{\max}}{2l^3}(3x^2l - x^3)$$

Free vibration of single degree of freedom system

Sol.



$$T_{\text{beam}} = \frac{1}{2} \int_0^l \frac{m}{l} \{ \dot{y}(x) \}^2 dx$$

$$\begin{aligned} T_{\text{max}} &= \frac{m}{2l} \left(\frac{\dot{y}_{\text{max}}}{2l^3} \right)^2 \int_0^l (3x^2l - x^3)^2 dx \\ &= \frac{1}{2} \frac{m}{l} \frac{\dot{y}_{\text{max}}^2}{4l^6} \left(\frac{33}{35} l^7 \right) = \frac{1}{2} \left(\frac{33}{140} m \right) \dot{y}_{\text{max}}^2 \end{aligned}$$

$$P = \frac{3EI}{l^3} y_{\text{max}} \Leftrightarrow F = KX \rightarrow K_{\text{eff}} = \frac{3EI}{l^3}$$

$$\omega_n = \sqrt{\frac{K_{\text{eff}}}{M + m_{\text{eff}}}} = \sqrt{\frac{3EI}{\left(M + \frac{33}{140} m \right) l^3}}$$